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Quadratic Optimisation with One Quadratic Equality Constraint

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ABSTRACT (U)

This report presents a theoretical framework for minimising a quadratic objective function subject to a quadratic equality constraint. The first part of the report gives a detailed algorithm which computes the global minimiser without calling special nonlinear optimisation solvers. The second part of the report shows how the developed theory can be applied to solve the time of arrival geolocation problem.

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Executive Summary

The theoretical work presented in this report is motivated by the need to solve many defence application problems such as the time of arrival geolocation problem. The mathematical tools needed to solve such a localisation problem, are developed in detail in the first part of the report (Part I) and are based on quadratic optimization with one quadratic equality constraint. The main contribution of this work is the development of an algorithm which provides a step-by-step procedure to solve the problem and address solution feasibility and uniqueness issues. This algorithm relies heavily on linear algebraic transformations and optimality condition properties to efficiently and exactly determine the problem minimiser. The localisation of an emitter source or a receiver based on time of arrival (TOA) measurements is a demonstration example given in the second part of the report (Part II), which illustrates how the developed theory is used.

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Hatem Hmam received his Ph.D. degree in Electrical Engineering from the University of Cincinnati, Ohio, USA, in 1992. He then pursued post-doctoral research in jet engine control at the University of Colorado, Boulder, and signal processing at Queensland University of Technology, Brisbane. He joined DSTO in 1996 and has worked in Weapons Systems and Electronic Warfare and Radar Divisions. He is currently a member of DEW group where he is developing and implementing passive localisation techniques of RF emitters.

Contents

1. Introduction

The current work was motivated by the need to solve nonconvex optimisation problems encountered in many engineering applications. This report focuses on developing an efficient algorithm to exactly solve the problem of minimising a multivariate quadratic cost function subject to a single quadratic equality constraint. The class of problems we would like to solve is

$$\begin{aligned} &\underset{z}{\text{minimize}}\ z^T P z + 2p^T z + u \\ &\text{subject to } z^T Q z + 2q^T z + v = 0 \end{aligned} \tag{I.1}$$

where the variable $z \in \mathbf{R}^n$ and $Q \in \mathbf{S}^n$, $(Q \neq 0)$. The parameters p and q are vectors of $\mathbf{R}^n$ while u and v are real numbers. The objective Hessian matrix P is assumed positive definite. An application where optimisation of this type arises is global positioning of a receiver using pseudo-range measurements. Section 6 of the report briefly demonstrates how this and related applications fit the general framework of (I.1) by applying the algorithm developed in this report to a time-of-arrival (TOA) localisation problem.

Variants of problem (I.1) appeared in the mathematical literature [3][4] but not with the intent to give a detailed algorithm to solve it. In fact, as we shall see later, advanced linear algebra techniques are used to transform (I.1) into a form where the problem can be solved efficiently and exactly. Fundamental results on the first and second order optimality conditions [3][4] help locate the global minimiser of (I.1).

This report is arranged as follows. Section 2 reformulates (I.1) using advanced linear algebra techniques. The purpose is to render optimisation problem (I.1) more tractable. Section 3 presents the notations used in this work. It also defines key matrices, eigenvalues and vectors that shape the behaviour of the optimisation problem. Section 4 addresses problem feasibility issues related to the constraint function. Section 5 presents optimality conditions and provides an algorithm that solves (I.1) efficiently and exactly. Section 6 forms the second part of the report and gives an illustrative geolocation example showing how the developed constrained optimisation algorithm is applied. Section 7 provides concluding remarks.

2. Problem Reformulation

To solve problem (I.1), we transform (I.1) into a more tractable problem as follows. Because P is positive definite, then there exists an invertible matrix, S, which simultaneously satisfies $SPS^T = I_n$ and $SQS^T = Q_D$ (diagonal) [7]. The matrix I_n is the n^{th} dimensional identity matrix. Using this matrix decomposition, problem (I.1) can be equivalently re-expressed as

$$\begin{aligned}&\underset{y}{\text{minimize}}\ y^T I_n y + 2b^T y + u \\ &\text{subject to } y^T Q_D y + 2\text{f}^T y + v = 0\end{aligned} \qquad \text{(I.2)}$$

where the parameters of (I.1) are related to those of (I.2) through $z = S^T y$, $b = Sp$ and $\text{f} = Sq$. Using the substitution, $x = y + b$, this equation can in turn, be transformed into

$$\begin{aligned}&\underset{x}{\text{minimize}}\ x^T I_n x + r \\ &\text{subject to } x^T Q_D x + 2c^T x + s = 0\end{aligned} \qquad \text{(I.2′)}$$

where $r = u - b^T b$, $c = \text{f} - Q_D b$ and $s = v + b^T Q_D b - 2\text{f}^T b$. Given that the minimiser of (I.2′) is not affected by the value of *r*, we may select $r = 0$, and therefore the basic optimisation problem to be solved reduces to

$$\begin{aligned}&\underset{x}{\text{minimize}}\ x^T x \\ &\text{subject to } h(x) = x^T Q_D x + 2c^T x + s = 0\end{aligned} \qquad \text{(I.3)}$$

The remainder of the report focuses on finding globally optimal solutions for problem (I.1) through solving (I.3). In particular if x^* is a global minimum solution of (I.3), then the minimiser of (I.1) is deduced as $z^* = S^T(x^* - Sp)$. Note that (I.3) can also be viewed as the problem of finding the closest point on a n-dimensional conic to the origin. Furthermore it should be noted that if $c = 0$ in (I.3), then only square terms exist in both the constraint and objective functions of (I.3). Solving such a problem becomes straightforward as (I.3) can be transformed into a linear program (LP). In the remaining part of the report, we consider the non-trivial problem where $c \neq 0$.

3. Some Useful Terminologies and Notations

In this section we introduce a number of notations, which prove useful in solving (I.3) or (I.1). These notations are related to Q_D and *c*. First note that if Q_D is negative semidefinite, then by multiplying the constraint of (I.3) by -1, the constraint remains unchanged but Q_D becomes positive semidefinite. Hence in the rest of the paper we won't consider negative semidefinite Q_D. By convention the diagonal matrix, Q_D, is represented as $\text{diag}(\mu_1, \cdots, \mu_l, \mu_{l+1} \cdots, \mu_m, 0 \cdots 0)$ where the *m* nonzero entries, μ_i, are arranged in a descending order as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l > 0 > \mu_{l+1} \geq \cdots \geq \mu_m$. Note that because Q_D is congruent to Q (i.e. $SQS^T = Q_D$ with S invertible) then by Sylvester's law of inertia [7], Q_D has the same number of positive, negative and zero eigenvalues as Q. We also define the largest positive and smallest negative eigenvalues of Q_D as $\mu_{\max} > 0$ and $\mu_{\min} < 0$. As an example if

$$Q_D = \begin{bmatrix} 4 & & & & \\ & 4 & & & \\ & & -1 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} \text{ and } c = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ 2 \end{bmatrix}$$

then $l = 2$, $m = 3$, $\mu_{\max} = 4$ and $\mu_{\min} = -1$. We also denote by $c_{\max}$ and $c_{\min}$ the subvectors of *c*, which correspond respectively to the extreme eigenvalues $\mu_{\max}$ and $\mu_{\min}$ of Q_D . Hence $c_{\max} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $c_{\min} = [4]$. Finally Q_D is singular in this example and therefore we let c_z be the component vector of *c*, associated with the zero eigenvalue of Q_D . This leads to $c_z = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ in this example.

4. Problem Feasibility

In this section we introduce a theorem detailing the conditions under which (I.3) is feasible.

Theorem 1.

1. If Q_D is indefinite, then the feasibility set of (I.3) is non-empty
2. If $Q_D \geq 0$, singular and $c_z \neq 0$, then the feasibility set of (I.3) is non-empty
3. If $Q_D > 0$, then the feasibility set of (I.3) is non-empty if and only if $s \leq \sum_{i=1}^{n} \frac{c_i^2}{\mu_i}$
4. If $Q_D \geq 0$ and $c_z = 0$, then the feasibility set of (I.3) is non-empty if and only if $s \leq \sum_{i=1}^{m} \frac{c_i^2}{\mu_i}$

Proof.

The constraint equation in (I.3) can be rearranged in the form $h(x) = \sum_{i=1}^{l} \mu_i x_i^2 + \sum_{i=l+1}^{m} \mu_i x_i^2 + 2 \sum_{i=m+1}^{n} c_i x_i + s$

where l corresponds to the number of positive eigenvalues (counting multiplicities), The second sum of squares corresponds to the negative eigenvalues of Q_D. The third term (if any) corresponds to the zero eigenvalues of Q_D.

1. If Q_D is indefinite then clearly $h(x)=0$ has always real solutions and hence feasible points always exist.

2. If $Q_D \geq 0$ and singular, then there are no negative eigenvalues (i.e. $m=l$) and therefore no second quadratic term appears in $h(x)$. If $c_z \neq 0$ then there exists at least one non-zero entry, c_k, in the vector, c_z. The associated variable, x_k, has no corresponding square term in $h(x)$. This means that by varying x_k and fixing the remaining variables, $h(x)$ can be made arbitrarily large (positive) or small (negative) and therefore by virtue of continuity of $h(x)$, solutions always exist to $h(x)=0$. Feasible points always exist under these conditions.

3. If $Q_D > 0$, then the minimum of $h(x)$ is $\eta = s - \sum_{i=1}^{n} \frac{c_i^2}{\mu_i}$. Hence feasible points exist if and only if $\eta \leq 0$, or in other words $s \leq \sum_{i=1}^{n} \frac{c_i^2}{\mu_i}$.

4. Similarly if $Q_D \geq 0$ and $c_z = 0$, then the minimum of $h(x)$ is $\eta = s - \sum_{i=1}^{m} \frac{c_i^2}{\mu_i}$ and therefore feasible points exist if and only if $s \leq \sum_{i=1}^{m} \frac{c_i^2}{\mu_i}$. □

Theorem 1 gives the conditions under which (I.3) is feasible. These, as explained below, also include uninteresting cases where constraint feasibility is restricted to a single point (i.e. when $Q_D > 0$ and $s = \sum_{i=1}^{n} \frac{c_i^2}{\mu_i}$) or to a subspace of reduced dimension (i.e. when $Q_D \geq 0$ singular, $c_z = 0$ and $s = \sum_{i=1}^{m} \frac{c_i^2}{\mu_i}$). In both these cases we have $\inf_{\mathbf{R}^n} h(x) = 0$.

The minimisation solution for these two special cases turns out to be trivial and unique as shown below.

1st case: $Q_D > 0$ and $s = \sum_{i=1}^{n} \frac{c_i^2}{\mu_i}$

The only feasible point is clearly, $x = -Q_D^{-1} c$, which is the global minimiser of (I.3).

2nd case: $Q_D \geq 0$ singular, $s = \sum_{i=1}^{m} \frac{c_i^2}{\mu_i}$ and $c_z = 0$

First we define the matrix Q_{Dnz} to be the submatrix of Q_D containing all nonzero diagonal entries and then we denote by c_{nz} the subvector of c corresponding to Q_{Dnz}. If we partition x as $x = \begin{bmatrix} x_{nz} \\ x_z \end{bmatrix}$, then clearly the point $x_{nz} = -Q_{Dnz}^{-1} c_{nz}$ satisfies the constraint equation with x_z arbitrary. It is straightforward to see that the minimiser of (I.3) then becomes $x^* = \begin{bmatrix} -Q_{Dnz}^{-1} c_{nz} \\ 0 \end{bmatrix}$.

To rule out these uninteresting rare cases, we also require that [3][4]

$$\inf_{\mathbf{R}^n} h(x) < 0 < \sup_{\mathbf{R}^n} h(x) \tag{I.4}$$

Condition (I.4) always holds for scenarios 1 and 2 of Theorem 1, but holds in scenarios 3 and 4 if and only if the inequalities are replaced with strict inequalities. In the rest of the report, we assume that problem feasibility holds in the sense of (I.4).

5. Finding the Global Minimum of (I.1)

Before solving (I.1) or equivalently (I.3) we need to be aware that the solution behaviour of (I.3) is affected by $c_{\max}$ and $c_{\min}$. Recall that these are the subvectors of c associated respectively with the largest positive and smallest negative eigenvalues of Q. Appendix A illustrate how $c_{\max} = 0$ or $c_{\min} = 0$ leads to solving a reduced optimisation problem, often with no unique solution. This observation has motivated the introduction of the next assumption.

Assumption 1.

Referring to problem formulation (I.3),

if $Q_D \geq 0$, then we assume that $c_{\max} \neq 0$,
if Q_D is indefinite, then we assume that $c_{\max} \neq 0$ and $c_{\min} \neq 0$.

This assumption gives a sufficient condition for the global minimum to be unique (a single point in $\mathbf{R}^n$) and helps to avoid the so-called "hard case" problem in [6]. Going back to the example given in the first part of this section, Q_D has both positive and negative eigenvalues. We clearly have $c_{\min} \neq 0$ but we don't have $c_{\max} \neq 0$. Hence Assumption 1 is not satisfied in this example.

Next we define the Lagrangian of problem (I.3) as

$$L(x,\lambda) = x^T I_n x - \lambda(x^T Q_D x + 2c^T x + s)$$

An optimal solution must satisfy the following first-order and constraint conditions

$$\begin{aligned} &(I_n - \lambda Q_D)x - \lambda c = 0 \\ &x^T Q_D x + 2c^T x + s = 0 \end{aligned} \tag{I.5}$$

From [3][4][5], the global minimum must also satisfy the second order optimality condition

$$I_n - \lambda Q_D \geq 0 \tag{I.6}$$

If λ^* is an optimal Lagrange multiplier satisfying (I.5) then from (I.6) we must have $D_\lambda = I_n - \lambda^* Q_D \geq 0$. The next proposition, helps us narrow down the region where the optimal Lagrange multiplier, λ^*, resides.

Proposition 1.

If $Q_D \geq 0$, then $\lambda^* \leq \lambda_{\max} = \dfrac{1}{\mu_{\max}}$.

If Q_D is indefinite, then $\dfrac{1}{\mu_{\min}} = \lambda_{\min} \leq \lambda^* \leq \lambda_{\max} = \dfrac{1}{\mu_{\max}}$.

Proof.

The proof mainly relies on the second order optimality condition $I_n - \lambda^* Q_D \geq 0$ at the global minimum. This condition leads to $1 - \lambda^* \mu_i \geq 0$ for all nonzero eigenvalues of Q_D . Hence the Lagrange multiplier region is given as the intersection of all intervals generated by each eigenvalue inequality ($1 - \lambda^* \mu_i \geq 0$). This necessarily means that the Lagrange multiplier region is an interval delimited, from one side or both, by the inverse of the largest positive or smallest negative eigenvalues of Q_D as indicated in the proposition. □

This proposition shows that the optimal Lagrange region is an interval of finite or infinite length. Let's denote this interval by $\overline{\Lambda} = [\lambda_{\min} \ \ \lambda_{\max}]$, where it should be understood that $\lambda_{\min}$ can be finite or infinite and $\lambda_{\min} < 0 < \lambda_{\max}$.

The first-order optimality condition in (I.5) reads as

$$\left(I_n - \lambda Q_D\right)x = \lambda c \tag{I.7}$$

Clearly given Assumption 1 and Proposition 1, it necessarily follows from (I.7) that the optimal Lagrange parameter, λ^* , cannot be $\lambda_{\min}$ or $\lambda_{\max}$. In other words λ^* must reside within the open interval $\Lambda = (\lambda_{\min} \;\; \lambda_{\max})$ and D_{λ^*} is invertible.

From (I.7), the globally optimal vector x can be obtained as a function of λ as

$$x = \lambda D_\lambda^{-1} c \tag{I.8}$$

Plugging (I.8) into the constraint $x^T Q_D\, x + 2c^T x + s = 0$ of (I.5) leads to

$$K(\lambda) = \lambda^2 c^T D_\lambda^{-T} Q_D D_\lambda^{-1} c + 2\lambda c^T D_\lambda^{-1} c + s = 0 \tag{I.9}$$

which reduces to

$$K(\lambda) = \sum_{i=1}^{m} \left[\frac{\lambda^2 \mu_i c_i^2}{(1-\lambda\mu_i)^2} + 2\frac{\lambda c_i^2}{(1-\lambda\mu_i)} \right] + 2\lambda \sum_{i=m+1}^{n} c_i^2 + s = 0 \tag{I.10}$$

or equivalently

$$K(\lambda) = \sum_{i=1}^{m} \frac{c_i^2}{\mu_i (1-\lambda\mu_i)^2} + 2\lambda \sum_{i=m+1}^{n} c_i^2 + \eta = 0 \tag{I.11}$$

where $\eta = s - \sum_{i=1}^{m} \frac{c_i^2}{\mu_i}$.

The linear term in (I.11) is present if and only if Q_D is singular and $c_z \neq 0$.

The next proposition examines the number of solutions of the secular equation (I.11) within the open interval Λ.

Proposition 2.

If

1. Q_D is indefinite, or
2. $Q_D \geq 0$ singular and $c_z \neq 0$, or
3. $Q_D \geq 0$ singular, $c_z = 0$ and $\eta < 0$, or
4. $Q_D > 0$ and $\eta < 0$,

then $K(\lambda)$ admits a unique finite solution, λ^*, in Λ. Furthermore the constraint function $K(\lambda)$ is increasing with λ.

Proof.

First we prove the existence of a solution λ in Λ to $K(\lambda)=0$. We will carry out this analysis by distinguishing whether Q_D is positive, indefinite, singular and so on.

If Q_D is indefinite, then from Assumption 1, we have $c_{\max} \neq 0$ and $c_{\min} \neq 0$ in (I.11). Also following Proposition 1, we have $\lambda_{\min} = \dfrac{1}{\mu_{\min}}$ and $\lambda_{\max} = \dfrac{1}{\mu_{\max}}$. Therefore

$$\left[\lim_{\lambda \to \lambda_{\min}^{+}} K(\lambda)\right] = -\infty \text{ and } \left[\lim_{\lambda \to \lambda_{\max}^{-}} K(\lambda)\right] = +\infty$$

If $Q_D \geq 0$ but singular, then we have $m < n$. From Assumption 1, we also have $c_{\max} \neq 0$ in (I.11). Using Proposition 1 it necessarily follows that $\lambda_{\min} = -\infty$ and $\lambda_{\max} = \dfrac{1}{\mu_{\max}}$. Hence

$$\left[\lim_{\lambda \to \lambda_{\min}^{+}} K(\lambda)\right] = \begin{cases} -\infty \text{ if } & c_z \neq 0 \\ \eta < 0 \text{ if } & c_z = 0 \end{cases} \text{ and } \left[\lim_{\lambda \to \lambda_{\max}^{-}} K(\lambda)\right] = +\infty$$

If $Q_D > 0$, then we have $m = n$ and $c_{\max} \neq 0$ in (I.11). From Proposition 1, $\lambda_{\min} = -\infty$ and $\lambda_{\max} = \dfrac{1}{\mu_{\max}}$) and it follows that

$$\left[\lim_{\lambda \to \lambda_{\min}^{+}} K(\lambda)\right] = \eta < 0 \text{ and } \left[\lim_{\lambda \to \lambda_{\max}^{-}} K(\lambda)\right] = +\infty$$

Clearly we conclude that in all cases $\left[\lim_{\lambda \to \lambda_{\min}^{+}} K(\lambda)\right] \times \left[\lim_{\lambda \to \lambda_{\max}^{-}} K(\lambda)\right] < 0$, and therefore by continuity of $K(\lambda)$, it necessarily follows that there must be at least one finite solution λ, in Λ, satisfying $K(\lambda)=0$.

Furthermore the derivative of $K(\lambda)$ with respect to λ, is $\dfrac{dK}{d\lambda} = 2\sum_{i=1}^{n} \dfrac{c_i^2}{(1-\lambda\mu_i)^3}$, which is clearly positive for all λ in Λ, implying that $K(\lambda)$ is increasing and therefore $K(\lambda)$ admits only one finite root, λ^*, in Λ.

□

Proposition 2 shows that the function $K(\lambda)$ is well behaved. Figure 1. shows one example of how $K(\lambda)$ might look like when the constraint function, Q_D, is indefinite. It is clear that irrespective of the size *n* of problem (I.3) or (I.1), efficient zero finding algorithms may be applied and are guaranteed to converge. Note also that $\lambda = 0$ is always within Λ, which may be used as a starting point for a Newton-based zero-finding algorithm. Furthermore if Q_D is indefinite (as in Figure 1.) then a bisection algorithm may be used to find the zero of

$K(\lambda)$. The starting interval for the bisection algorithm is $[\lambda_{\min} + \varepsilon \;\; \lambda_{\max} - \varepsilon] \subset \Lambda$, where $\varepsilon > 0$, small enough to be able to evaluate $K(\lambda)$ at the first iteration. To speed up convergence, a switch to a faster zero finding algorithm may be enabled after a few iterations of the bisection algorithm. If however, Q_D is positive semidefinite then $K(\lambda)$ is convex within $\Lambda = (-\infty \;\; \lambda_{\max})$ (because the second derivative $\dfrac{d^2K}{d\lambda^2} = 6\sum_{i=1}^{n} \dfrac{\mu_i c_i^2}{(1-\lambda\mu_i)^4}$ is positive in Λ) and the zero-finding algorithm may be tailored to take advantage of the convex property.

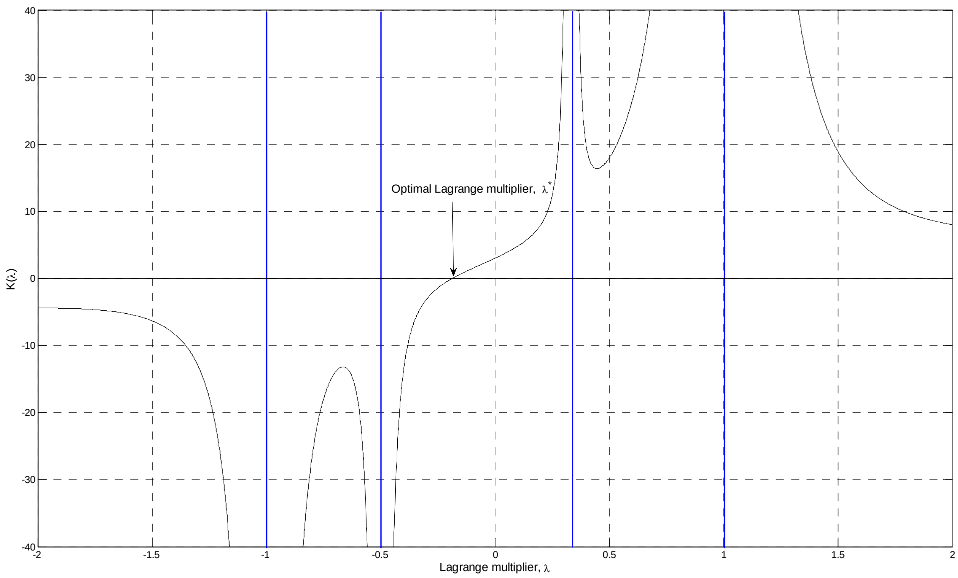

Figure 1: An illustrative example of how $K(\lambda)$ might look. The optimal Lagrange multiplier is within $\Lambda = (-1/2 \;\; 1/3)$

The algorithm to solve (I.1) is summarised below:

1. Ensure that Hessian matrix, P, of the objective function of (I.1) is positive definite. Otherwise stop
2. Transform problem (I.1) into problem (I.3) using standard simultaneous matrix decomposition. If $Q_D \le 0$, multiply the constraint equation by -1. The new Q_D becomes -Q_D, which is positive semidefinite.
3. Examine the feasibility of (I.3) according to Theorem 1 and property (I.4). If (I.3) is not feasible then stop. If $Q_D > 0$ and $s = \sum_{i=1}^{n} \frac{c_i^2}{\mu_i}$, or if $Q_D \ge 0$ singular, $c_z = 0$ and $s = \sum_{i=1}^{m} \frac{c_i^2}{\mu_i}$, then carry out the special solving procedure described in Section I.2 and then stop.
4. Check Assumption 1. If it holds, determine the interval, Λ, where the Lagrange multiplier resides. If Assumption 1 is not met stop.
5. Apply a zero finding algorithm to (I.11) within Λ to compute λ^*.
6. Plug in the computed λ^* in (I.8) to determine the globally optimal solution, x^*. Note that because $I_n - \lambda^* Q_D$ in (I.8) is diagonal, inverting this matrix is straightforward.
7. Compute the global minimiser of problem (I.1) using $z^* = S^T(x^* - Sp)$
8. Stop

6. Localisation of an Emitter Using TOA Measurements

In the second part of the report we introduce an application, which motivated the mathematical analysis presented in part I. The localisation of an emitter source or a receiver based on time of arrival (TOA) measurements [8], has long been of interest both in civilian and military applications (e.g., GPS). An objective function that links TOA measurements to the emitter position in the plane and transmit time, t, may be given as [9]

$$p(x,y,t)=\sum_{i=1}^{i=N}\omega_i\left((x-x_i)^2+(y-y_i)^2-v^2(t_i-t)^2\right)^2 \qquad \text{(II.1)}$$

where $(x, y)^T$ stands for the emitter's position to be estimated and $(x_i, y_i)^T$ the known two-dimensional coordinates of the i^{th} receiver. The parameters t_i are the measured time of arrivals. The parameters ω_i are positive constant weights and v is the speed of light (or speed of sound for an acoustic source). An estimate of the emitter position and transmit time is obtained by finding the global minimum of the unconstrained cost function (II.1), which is a quartic function of three variables. This quartic cost function is nonconvex and hence classical methods such SDP (Semidefinite Programming [1][2]) based relaxation algorithms may not yield the exact global minimiser and are generally slower to execute.

If we put $r_i - r = v(t_i - t)$ then (II.1) can be written more compactly as

$$p(x,y,r)=\sum_{i=1}^{i=N}w_i\left((x-x_i)^2+(y-y_i)^2-(r_i-r)^2\right)^2 \qquad \text{(II.2)}$$

If we expand (II.2) and put

$$2e = x^2 + y^2 - r^2 \qquad \text{(II.3)}$$

in the polynomial objective function (II.2) then (II.2) reduces to the four variable cost function

$$p(x,y,r,e)=\sum_{i=1}^{i=N}w_i p_i^2(x,y,r,e) \qquad \text{(II.4)}$$

where $p_i(x,y,r,e)=2\left(e-xx_i-yy_i+rr_i+\dfrac{\beta_i}{2}\right)$ is linear in its four arguments. The parameter β_i is defined as $\beta_i = x_i^2 + y_i^2 - r_i^2$. By grouping the quadratic and linear terms of (II.2) and (II.4), then an estimate of the emitter position and range bias, r, can be obtained by solving the equivalent constrained minimisation problem

$$\begin{aligned}&\min_z z^T P\, z + 2 p^T z + d\\ &\text{subject to } z^T Q\, z + 2 q^T z = 0\end{aligned} \qquad \text{(II.5)}$$

where $z=(x,y,r,e)^T \in \mathbf{R}^4$ and P and p store the coefficients of the quadratic and linear terms of (II.4). In fact as shown in [9], if one defines the i[th] compound measurement vector u_i as $u_i = 2(-x_i,-y_i,r_i,1)^T$ then it follows that

$$\begin{cases} P & = & \sum_{i=1}^{N} w_i \left(u_i u_i^T\right) \\ p & = & \sum_{i=1}^{N} w_i \beta_i u_i \\ d & = & \sum_{i=1}^{N} w_i \beta_i^2 \end{cases} \tag{II.6}$$

where d is a constant in the objective function and therefore can be ignored.
Also from (II.3) it clearly follows that Q is diag(1,1,-1,0) and $q=(0,0,0,-1)^T$.

Solving Problem (II.6)

As explained in Part I of the report, problem (II.1) can be transformed into (I.3). Solving the optimisation problem (II.5) then amounts to applying the algorithm of Section 5. Some of the steps of the algorithm call for checking the constraint feasibility of (II.5) and whether the assumptions given in Part I are satisfied.

First we note that since Q is indefinite then Q_D is indefinite and hence the feasibility set is non-empty and (I.4) is satisfied. Also given N ($N \geq 4$) receivers located in general positions (e.g., not accidentally aligned), then P is positive definite.

Simulation Results

Computer simulation using MATLAB have been conducted to evaluate the performance of the TOA-based geolocation. Comparisons were made with a similar localisation technique based on time-difference-of-arrival (TDOA) measurements. The TDOA-based localisation method is selected from [9, Section 3.1] and is the constrained weighted least square error (CWLS) method. This localisation technique requires solving a quartic equation in the Lagrange multiplier, η, (refer to (29) in [10]). The localisation solution is given by evaluating (28) in [10]. The TDOA measurement error covariance matrix used is

$$C = \frac{\sigma_{TDOA}^2}{2}\begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ 1 & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix}$$

where a common TOA variance error is assumed. The main difference between the TDOA and the TOA-based geolocation technique is the estimation of the additional parameter, range bias (or transmit time).

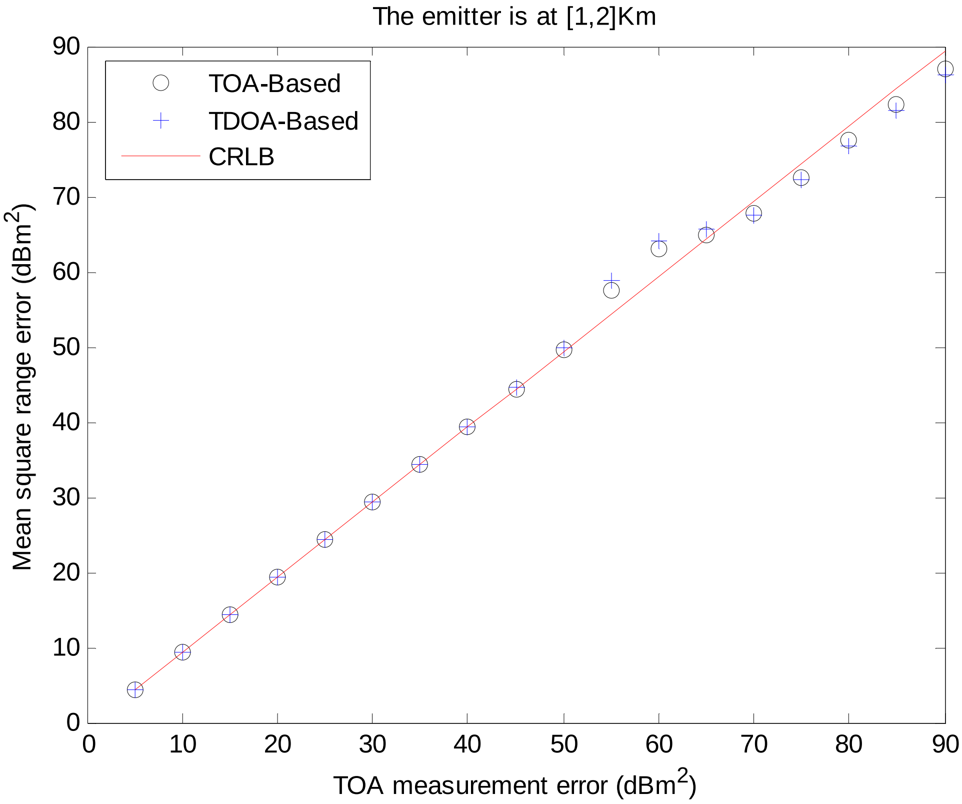

Figure 2: Mean square range error as a function of measurement error for both TOA and TDOA constraint-based localisation

However, it is emphasized in [11] that both localisation techniques are equivalent if accurate measurement error statistics were used for both methods and that performance variations are mainly due to inaccurate knowledge of measurement error statistics or simplified implementations. As can be seen in Figure 2, the performance of both constrained estimation methods approached the corresponding Cramer-Rao lower bound (CRLB) for a wide range of measurement errors. We assume in this figure that the TOA measurement error is normally distributed and that 5 stationary receivers are available and are positioned respectively at $(3,0)^T$, $(-2,1)^T$, $(0,-2)^T$, $(1,3)^T$ and $(3,3)^T$. The emitter or source is located at $(1,2)^T$ and all distances are expressed in Km. The errors on both axes in the figure are displayed in dBm^2. In particular the y-axis of the figure gives the mean square distance error between the estimated and the true position.

7. Conclusions

This report lays down a theoretical framework for minimising a quadratic objective function subject to a quadratic equality constraint. We assume that the Hessian of the objective function is positive definite. A detailed algorithm is presented, which computes the minimiser in a straightforward manner, despite the problem non-convexity. Most importantly this algorithm finds the global minimiser exactly, subject only to computer hardware precision, and does not require special solvers based on linear matrix inequalities (LMI) or SDP relaxation.

This work is motivated by the need to solve many engineering applications such as time of arrival localisation. The localisation of an emitter source based on time of arrival (TOA) measurements, was the focus study of Part II (Section 6) in this report. It is shown in detail how the problem is converted into a quadratic minimisation problem with one quadratic equality constraint. The global solution follows from the developed theory of Section 5. Finally we should point out that global minimisation problems involving more than one quadratic constraint are extremely hard to solve [3][5] and until today no sure techniques exist that can handle more than one quadratic constraint.

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Appendix A:

The main point of this appendix is to show that if $c_{\max} = 0$ or $c_{\min} = 0$ then the problem dimension is reduced from n to n' with $n' < n$. To easily demonstrate this we assume that only $c_{\max} = 0$. We define Q_R to be the submatrix of Q_D, which excludes the eigenvalue $\mu_{\max}$. Clearly the constraint equation of (I.3) can be expressed as

$$x_1^2 + \cdots + x_j^2 = -\frac{1}{\mu_{\max}}\left[x_R^T Q_R\, x_R + 2c_R^T x_R + s\right] \qquad \text{(A.1)}$$

where the index j ($j \geq 1$) stands for the multiplicity of the eigenvalue, $\mu_{\max}$, and c_R is the subvector of c corresponding to Q_R. The vector x_R consists of the remaining $(n-j)$ elements of x. Note that the diagonal matrix, $\frac{1}{\mu_{\max}} Q_R$, has all its diagonal terms less than 1 and therefore $D_R = I_{n-j} - \frac{1}{\mu_{\max}} Q_R$ is positive definite. The objective function, which can now be expanded as

$$x^T x = (x_1^2 + \cdots + x_j^2) + x_R^T x_R$$

becomes after substitution of (A.1) $x_R^T D_R\, x_R - \frac{2}{\mu_{\max}} c_R^T x_R - \frac{s}{\mu_{\max}}$. Hence the new minimisation problem reduces to

$$\begin{aligned} &\underset{x_R}{\text{minimize}} \quad x_R^T D_R\, x_R - \frac{2}{\mu_{\max}} c_R^T x_R - \frac{s}{\mu_{\max}} \\ &\text{subject to} \quad x_R^T Q_R\, x_R + 2c_R^T x_R + s \leq 0 \end{aligned} \qquad \text{(A.2)}$$

where the inequality constraint in (A.2) is the result of the right hand side of (A.1) being necessarily non-negative.

Now if the unconstrained solution of (A.2) satisfies the constraint $x_R^T Q_R\, x_R + 2c_R^T x_R + s \leq 0$, then the minimiser of (I.3) consists of all vectors $x = \begin{bmatrix} x_{1\cdots j} \\ x_R \end{bmatrix}$, where x_R is the unconstrained solution of (A.2) and $x_{1\cdots j} = (x_1, \cdots, x_j)^T$ satisfying (A.1). It clearly follows that if the strict inequality $x_R^T Q_R\, x_R + 2c_R^T x_R + s < 0$ holds then multiple solutions exist. If the unconstrained solution is not feasible, then the inequality constraint must be active (i.e. become an equality constraint) and the optimisation problem then reads as

$$\begin{aligned} &\underset{x}{\text{minimize}} \quad x_R^T D_R\, x_R - \frac{2}{\mu_{\max}} c_R^T x_R - \frac{s}{\mu_{\max}} \\ &\text{subject to} \quad x_R^T Q_R\, x_R + 2c_R^T x_R + s = 0 \end{aligned} \tag{A.3}$$

But problem (A.3) is very similar to (I.1) with D_R positive definite as required by Assumption 1. The minimiser of (A.3) is now, however, sought in the reduced space $\mathbf{R}^{n-j}$, instead of $\mathbf{R}^n$.

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19. ABSTRACT

This report presents a theoretical framework for minimising a quadratic objective function subject to a quadratic equality constraint. The first part of the report gives a detailed algorithm which computes the global minimiser without calling special nonlinear optimisation solvers. The second part of the report shows how the developed theory can be applied to solve the time of arrival geolocation problem.

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